Bracket Notations
Part-I

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Bracket Notations

1. Also Known as Dirac Notation because Physicist Paul Dirac has introduced it.
2. It is a compact and convenient way of representing quantum states and operations in quantum computing.
3. It is widely used because it simplifies complex mathematics of quantum theory.

$$\langle Bra| \; |ket\rangle$$

Different Forms of Bracket Notations

Ar. No.	Name	Symbol	Purpose
1	Ket Notations	$\|.\rangle$	Column Vector
2	Bra Notations	$\langle.\|$	Row Vector
3	Ket-Ket Notations	$\|.\rangle\|.\rangle$	Tensor Product of Column Vector
4	Bra-Bra Notations	$\langle.\|\langle.\|$	Tensor Product of Row Vector
5	Ket-Bra Notations	$\|.\rangle\langle.\|$	Matrix Representation
6	Bra-Ket Notations	$\langle.\|\|.\rangle$	Inner Product

(1) Ket Notations

Representing Quantum States through Column Vectors

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Converting Ket to Column Vectors

$$|11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad |101\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$D = 2^n$ dimensional column vector represents n bit information.

(1) Ket Notations

Representing Column Vectors through Ket notations

Example 1

$$\begin{pmatrix} 7 \\ 3+5i \end{pmatrix} = 7|0\rangle + (3+5i)\,|1\rangle$$

The basis used is $\{|0\rangle\,, |1\rangle\}$

Example 2

$$\begin{pmatrix} 7 \\ 0 \\ i+3 \\ 0 \end{pmatrix} = 7|00\rangle + (i+3)\,|10\rangle$$

The basis used is $\{|00\rangle\,, |01\rangle\,, |10\rangle\,, |11\rangle\}$

(2) Bra Notations

Representing Row Vectors through Bra notations

$\langle 0|$

$\langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix}$

$\langle 1|$

$\langle 1| = \begin{pmatrix} 0 & 1 \end{pmatrix}$

The ket and bra notations are interconvertible.
$|\psi\rangle^+ = \langle\psi|$
$\langle\psi|^+ = |\psi\rangle$

Example in bra notations

$\langle\psi| = (3 - 5i)\langle 0| + 7\langle 1|$

$\begin{pmatrix} 3 - 5i & 7 \end{pmatrix}$

Example in bra notations

$|\psi\rangle = (3 + 5i)|0\rangle + 7|1\rangle$

$\begin{pmatrix} 3 + 5i \\ 7 \end{pmatrix}$

(2) Bra Notations

Representing Row Vectors through Bra notations

Example 2

$\begin{pmatrix} 3 & 0 & i & 7 \end{pmatrix} =$
$3\langle 00| + i\,\langle 10| + 7\,\langle 11|$

Here, the basis is
$\{\langle 00|\,,\langle 01|\,,\langle 10|\,,\langle 11|\}$

(3) Ket-Ket Notations

It represents the tensor product of Column Vectors

What is a tensor product?

Example: Two matrices A and B

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \end{pmatrix}$$

Tensor Product of A and B

$$A \otimes B = \begin{pmatrix} 1.B & 0.B \\ 0.B & 2.B \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 4 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 & 8 & 14 \end{pmatrix}$$

$$|\psi\rangle \otimes |\theta\rangle = |\psi\rangle\,|\theta\rangle = |\psi\theta\rangle$$

(3) Ket-Ket Notations

Example

Two Given Functions

$\psi = i\,|0\rangle + 7\,|1\rangle$

$\phi = |00\rangle + 3\,|10\rangle + 7\,|11\rangle$

Vector Representations

$$|\psi\rangle = \begin{pmatrix} i \\ 7 \end{pmatrix}$$

$$|\phi\rangle = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 7 \end{pmatrix}$$

Tensor product: $|\psi\phi\rangle$

$|\psi\phi\rangle = i\,|000\rangle + 3i\,|010\rangle + 7i\,|011\rangle + 7\,|100\rangle + 21\,|110\rangle + 49\,|111\rangle$

Tensor product: $|\psi\phi\rangle$

$$|\psi\phi\rangle = \begin{pmatrix} i.\phi \\ 7.\phi \end{pmatrix} = \begin{pmatrix} i \\ 0 \\ 3i \\ 7i \\ 7 \\ 0 \\ 21 \\ 49 \end{pmatrix}$$