

Quantum Gates
One-Inputs
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Quantum Gate Properties

- ▶ Quantum gates are **unitary matrices** transforming inputs to outputs

$$U|\alpha\rangle = |\beta\rangle$$

- ▶ A **unitary matrix** means

$$UU^+ = U^+U = I$$

- ▶ Some quantum gates or unitary matrices are **Hermitian matrices**.

$$H^+ = H$$

- ▶ For a matrix that is both unitary and Hermitian

$$A = A^{-1} \text{ or } A.A^{-1} = I$$

such quantum gates are **reversible quantum gates**.

An Example

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Y^+ = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$YY^+ = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \times 0 + -ixi & 0 \times -i + -ix0 \\ ix0 + 0xi & ix - i + 0 \times 0 \end{pmatrix}$$

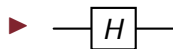
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= I$$

Pauli X Gate

- ▶ This is the Unitary and Hermitian Matrix

The symbol is



The matrix is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- ▶ The working is-

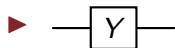
1. $X|0\rangle = |1\rangle$
2. $X|1\rangle = |0\rangle$
3. $X(\alpha|0\rangle + \beta|1\rangle) = \alpha X|0\rangle + \beta X|1\rangle = \alpha|1\rangle + \beta|0\rangle$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \times \alpha + 1 \times \beta \\ 1 \times \alpha + 0 \times \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \beta|0\rangle + \alpha|1\rangle$$

Pauli Y Gate

- ▶ This is the Unitary and Hermitian Matrix

The symbol is



The matrix is

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

- ▶ The working is-

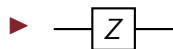
1. $Y|0\rangle = i|1\rangle$
2. $Y|1\rangle = -i|0\rangle$
3. $Y(\alpha|0\rangle + \beta|1\rangle) = \alpha Y|0\rangle + \beta Y|1\rangle = \alpha i|1\rangle - \beta i|0\rangle$

$$Y|0\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \times 1 + -i \times 0 \\ i \times 1 + 0 \times 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = i|1\rangle$$

Pauli Z Gate

- ▶ This is the Unitary and Hermitian Matrix

The symbol is



The matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- ▶ The working is-

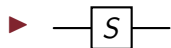
1. $Z|0\rangle = |0\rangle$
2. $Z|1\rangle = -|1\rangle$
3. $Z(\alpha|0\rangle + \beta|1\rangle) = \alpha Z|0\rangle + \beta Z|1\rangle = \alpha|0\rangle - \beta|1\rangle$

$$Z|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \times 0 + 0 \times 1 \\ 0 \times 0 + -1 \times 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -|1\rangle$$

S Gate

- ▶ This is the Unitary but **not Hermitian** Matrix

The symbol is



The matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

- ▶ The working is-

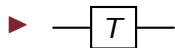
1. $S|0\rangle = |0\rangle$
2. $S|1\rangle = i|1\rangle$
3. $S(\alpha|0\rangle + \beta|1\rangle) = \alpha S|0\rangle + \beta S|1\rangle = \alpha|0\rangle + i\beta|1\rangle$

$$S|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \times 0 + 0 \times 1 \\ 0 \times 0 + i \times 1 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = i|1\rangle$$

T Gate

- ▶ This is the Unitary but **not Hermitian** Matrix

The symbol is



The matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{\pi/4} \end{pmatrix}$$

- ▶ The working is-

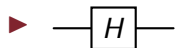
1. $T|0\rangle = |0\rangle$
2. $T|1\rangle = e^{\pi/4}|1\rangle$
3. $T(\alpha|0\rangle + \beta|1\rangle) = \alpha T|0\rangle + \beta T|1\rangle = \alpha|0\rangle + \beta e^{\pi/4}|1\rangle$

$$T|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & e^{\pi/4} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \times 0 + 0 \times 1 \\ 0 \times 0 + e^{\pi/4} \times 1 \end{pmatrix} = \begin{pmatrix} 0 \\ e^{\pi/4} \end{pmatrix} = e^{\pi/4}|1\rangle$$

Hadamard Gate

- ▶ This is the Unitary and Hermitian Matrix

The symbol is



The matrix is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- ▶ The working is-

1. $H|0\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$

2. $H|1\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$

3. $H(\alpha|0\rangle + \beta|1\rangle) = \alpha H|0\rangle + \beta H|1\rangle = \alpha \frac{|0\rangle+|1\rangle}{\sqrt{2}} + \beta \frac{|0\rangle-|1\rangle}{\sqrt{2}}$

$$H|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \times 0 + 1 \times 1 \\ 1 \times 0 + -1 \times 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$$