# Quantum Gates
## One-Inputs

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# Quantum Gate Properties

- Quantum gates are unitary matrices transforming inputs to outputs
$$U|\alpha\rangle = |\beta\rangle$$
- A unitary matrix means
$$UU^{+} = U^{+}U = I$$
- Some quantum gates or unitary matrices are Hermitian matrices.
$$H^{+} = H$$
- For a matrix that is both unitary and Hermitian
$$A = A^{-1} or A.A^{-1} = I$$

such quantum gates are reversible quantum gates.

# An Example

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Y^{+} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

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$$YY^{+} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0x0 + -ixi & 0x - i + -ix0 \\ ix0 + 0xi & ix - i + 0x0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= I$$

# Pauli X Gate

- This is the Unitary and Hermitian Matrix

The symbol is

- 


The matrix is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- The working is-
  1. $X|0\rangle = |1\rangle$
  2. $X|1\rangle = |0\rangle$
  3. $X(\alpha|0\rangle + \beta|1\rangle) = \alpha X|0\rangle + \beta X|1\rangle = \alpha|1\rangle + \beta|0\rangle$

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$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0x\alpha + 1x\beta \\ 1x\alpha + 0x\beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \beta|0\rangle + \alpha|1\rangle$$

# Pauli Y Gate

- This is the Unitary and Hermitian Matrix

The symbol is

- 


The matrix is

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

- The working is-
  1. $Y|0\rangle = i|1\rangle$
  2. $Y|1\rangle = -i|0\rangle$
  3. $Y(\alpha|0\rangle + \beta|1\rangle) = \alpha Y|0\rangle + \beta Y|1\rangle = \alpha i|1\rangle - \beta i|0\rangle$

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$$Y|0\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0x1 + -ix0 \\ ix1 + 0x0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = i|1\rangle$$

# Pauli Z Gate

- This is the Unitary and Hermitian Matrix

The symbol is

- 


The matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The working is-
  1. $Z\left|0\right\rangle = \left|0\right\rangle$
  2. $Z\left|1\right\rangle = -\left|1\right\rangle$
  3. $Z(\alpha\left|0\right\rangle + \beta\left|1\right\rangle) = \alpha Z\left|0\right\rangle + \beta Z\left|1\right\rangle = \alpha\left|0\right\rangle - \beta\left|1\right\rangle$

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$$Z\left|1\right\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1x0 + 0x1 \\ 0x0 + -1x1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\left|1\right\rangle$$

# S Gate

- This is the Unitary but not Hermitian Matrix

The symbol is

- 


The matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

- The working is-
  1. $S|0\rangle = |0\rangle$
  2. $S|1\rangle = i|1\rangle$
  3. $S(\alpha|0\rangle + \beta|1\rangle) = \alpha S|0\rangle + \beta S|1\rangle = \alpha|0\rangle + i\beta|1\rangle$

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$$S|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1x0 + 0x1 \\ 0x0 + ix1 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = i|1\rangle$$

# T Gate

- This is the Unitary but not Hermitian Matrix

The symbol is

- 


The matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{\pi/4} \end{pmatrix}$$

- The working is-
  1. $T|0\rangle = |0\rangle$
  2. $T|1\rangle = e^{\pi/4}|1\rangle$
  3. $T(\alpha|0\rangle + \beta|1\rangle) = \alpha T|0\rangle + \beta T|1\rangle = \alpha|0\rangle + \beta e^{\pi/4}|1\rangle$

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$$T|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & e^{\pi/4} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1x0 + 0x1 \\ 0x0 + e^{\pi/4}x1 \end{pmatrix} = \begin{pmatrix} 0 \\ e^{\pi/4} \end{pmatrix} = e^{\pi/4}|1\rangle$$

# Hadamard Gate

- This is the Unitary and Hermitian Matrix

The symbol is

- 


The matrix is

$$1/\sqrt{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

- The working is-
  1. H$|0\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$
  2. H$|1\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$
  3. $H(\alpha\,|0\rangle + \beta\,|1\rangle) = \alpha H\,|0\rangle + \beta H\,|1\rangle = \alpha\frac{|0\rangle+|1\rangle}{\sqrt{2}} + \beta\frac{|0\rangle-|1\rangle}{\sqrt{2}}$

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$$H\,|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1x0+1x1\\ 1x0+-1x1\end{pmatrix} = \begin{pmatrix}1\\ -1\end{pmatrix} = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$$