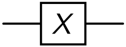
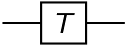
Quantum Gates
Exercises

Arvind W Kiwelekar

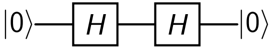
Summary of Single Bit Quantum Gates

Symbol	Application	Matrix
	$X\|0\rangle = \|1\rangle$ $X\|1\rangle = \|0\rangle$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
	$Y\|0\rangle = i\|1\rangle$ $Y\|1\rangle = -i\|0\rangle$	$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$
	$Z\|0\rangle = \|0\rangle$ $Z\|1\rangle = -\|1\rangle$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Summary of Single Bit Quantum Gates

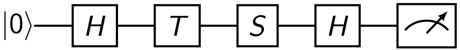
Symbol	Application	Matrix
	$S\|0\rangle = \|1\rangle$ $S\|1\rangle = i\|0\rangle$	$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$
	$T\|0\rangle = \|0\rangle$ $T\|1\rangle = e^{i/4}\|1\rangle$	$\begin{pmatrix} 1 & -0 \\ 0 & e^{i/4} \end{pmatrix}$
	$H\|0\rangle = \frac{\|0\rangle + \|1\rangle}{\sqrt{2}}$ $H\|1\rangle = \frac{\|0\rangle - \|1\rangle}{\sqrt{2}}$	$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Reversible Gates



$$
\begin{aligned}
& H\frac{|0\rangle+|1\rangle}{\sqrt{2}} \\
&= \frac{H|0\rangle+H|1\rangle}{\sqrt{2}} \\
&= \frac{H|0\rangle}{\sqrt{2}} + \frac{H|1\rangle}{\sqrt{2}} \\
&= \frac{|0\rangle+|1\rangle}{\sqrt{2}\sqrt{2}} + \frac{|0\rangle-|1\rangle}{\sqrt{2}\sqrt{2}} \\
&= \frac{|0\rangle+|1\rangle+|0\rangle-|1\rangle}{2} \\
&= \frac{2|0\rangle}{2} \\
&= |0\rangle
\end{aligned}
$$

Quantum Circuit



$$HSTH|0\rangle = HST\frac{|0\rangle+|1\rangle}{\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}}HS(T|0\rangle + T|1\rangle)$$
$$= \frac{1}{\sqrt{2}}HS(|0\rangle + e^{i\pi/4}|1\rangle)$$
$$= \frac{1}{\sqrt{2}}H(S|0\rangle + e^{i\pi/4}S|1\rangle)$$
$$= \frac{1}{\sqrt{2}}H(|0\rangle + ie^{i\pi/4}|1\rangle)$$
$$= \frac{1}{\sqrt{2}}(H|0\rangle + ie^{i\pi/4}H|1\rangle)$$
$$= \frac{1}{\sqrt{2}}(\frac{|0\rangle+|1\rangle}{\sqrt{2}} + ie^{i\pi/4}\frac{|0\rangle-|1\rangle}{\sqrt{2}})$$

$$(1 + ie^{i/\pi/4})|0\rangle+(1 - ie^{i/\pi/4})|1\rangle\ /2$$

$$P(|0\rangle = (1 + ie^{i/\pi/4})^2$$
$$= 0.146$$
$$P(|1\rangle = (1 - ie^{i/\pi/4})^2$$
$$= 0.854$$

Example Problem

Consider an operator U perform the following mapping on the Z-basis states.

$$U|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \quad U|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

(a) What is the U as matrix

(b) What is U$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

(c) From your answer tell whether U is a valid quantum gate

Example Problem

(a) What is the U as matrix U=

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}$$

$$U^{+} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}$$

$$UU^{+} = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1x1 + -ixi & 1xi + -ix1\\ -ix1 + 1xi & -ixi + 1x1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = I$$

Solution

(b) What is $U\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$

$$
\begin{aligned}
&U(\alpha|0\rangle + \beta|1\rangle) \\
&= \alpha U|0\rangle + \beta U|1\rangle) \\
&= \frac{\alpha(|0\rangle - i|1\rangle)}{\sqrt{2}} + \frac{\beta(-i|0\rangle + |1\rangle)}{\sqrt{2}} \\
&= \frac{(\alpha - i\beta)|0\rangle + (\beta - i\alpha)|1\rangle}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix}\alpha - i\beta \\ \beta - i\alpha\end{pmatrix}
\end{aligned}
$$

Conclusion

- A quantum bit or qubit represents three different states.
- Bracket notation is a powerful notation that is used for quantum computing to simplify complex vector computations.
- Quantum gates are basic building blocks around which a complex quantum circuit can be built.

Solution

(b) What is U$\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$

$$\mathrm{U}(\alpha\,|0\rangle + \beta\,|1\rangle)$$
$$= \alpha U\,|0\rangle + \beta U\,|1\rangle)$$
$$= \frac{\alpha(|0\rangle - i|1\rangle)}{\sqrt{2}} + \frac{\beta(-i|0\rangle + |1\rangle)}{\sqrt{2}}$$
$$= \frac{(\alpha - i\beta)|0\rangle + (\beta - i\alpha)|1\rangle}{\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix}\alpha - i\beta \\ \beta - i\alpha\end{pmatrix}$$